18.152 PROBLEM SET 5 SOLUTIONS

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1. Problem 3

Most of students figured this problem out. Some of you made small arithmetic errors in your p-sets. The solution below is based on the work of Eli Garcia

Solution to Problem 3. We rewrite the equation as

$$(\partial_t - 2\partial_x)(\partial_t + \partial_x)u = 0.$$

Let $v = (\partial_x + \partial_t)u$, then

$$(\partial_t - 2\partial_x)v(x,t) = 0.$$

This means that $v(x,t) = G(x+2t)$ for some smooth function $G : \mathbb{R} \to \mathbb{R}$. Note that

$$\begin{aligned} u(x,t) - u(x-t,0) &= \int_0^t \partial_s[u(x-t+s,s)]ds, \\ &= \int_0^t v(x-t+s,s)ds, \\ &= \int_0^t G(x-t+3s)ds. \end{aligned}$$

Now consider the boundary condition

$$\begin{aligned} u(x,0) &= g(x), \\ u_t(x,0) &= h(x) \text{ when } t = 0, \end{aligned}$$

so

$$u(x,t) = g(x-t) + \int_0^t G(x-t+3s)ds.$$

and

$$\begin{aligned} h(x) = u_t(x,0) &= -g'(x) + \Big[- \int_0^t G'(x-t+3s)ds + G(x+2t) \Big]\Big|_{t=0} \\ &= -g'(x) + G(x). \end{aligned}$$

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As a result, $G(x) = g'(x) + h(x)$. In conclusion, we have

$$\begin{aligned} u(x,t) &= g(x-t) + \int_0^t g'(x-t+3s) + h(x-t+3s)ds, \\ &= g(x-t) + \frac{1}{3}(g(x+2t) - g(x-t)) + \int_0^t h(x-t+3s)ds, \\ &= \frac{1}{3}g(x+2t) + \frac{2}{3}g(x-t) + \frac{1}{3}\int_{x-t}^{x+2t} h(y)dy. \end{aligned}$$ □

2. Problem 4(b)

Most of you figured this problem out. Here is a side remark: whenever you are asked to find a counter example, try to find the simplest one to make things concrete. The following example is provided by the work of Eli Garcia.

Proof. For simplicity, take $L = 1$ and consider the function:

$$\begin{aligned} u : [0,1]_s \times [0,\infty)_t &\to \mathbb{R} \\ u(x,t) &= \sin(2\pi x)\sin(2\pi t). \end{aligned}$$

Then u solves the wave equation with Dirichlet boundary condition:

$$\begin{aligned} u_{tt} &= u_{xx},\ (x,t) \in [0,1]_s \times [0,\infty)_t, \\ u(0,t) &= u(1,t) = 0, \\ u(x,0) &= 0, \\ u_t(x,0) &= 2\pi\sin(2\pi x). \end{aligned}$$

Finally,

$$\begin{aligned} \int_{[0,1]} |u_x|^2 &= 4\pi^2 \int_{[0,1]} |\cos(2\pi x)\sin(2\pi t)|^2 = 2\pi^2|\sin(2\pi t)|^2, \\ \int_{[0,1]} |u_t|^2 &= 4\pi^2 \int_{[0,1]} |\sin(2\pi x)\cos(2\pi t)|^2 = 2\pi^2|\cos(2\pi t)|^2. \end{aligned}$$

So the sum

$$\int_{[0,1]} |u_t|^2 + |u_x|^2 = 2\pi^2.$$

is a constant function in t. It oscillates between $\int_{[0,1]} |u_x|^2$ and $\int_{[0,1]} |u_t|^2$ as the time evolves. □